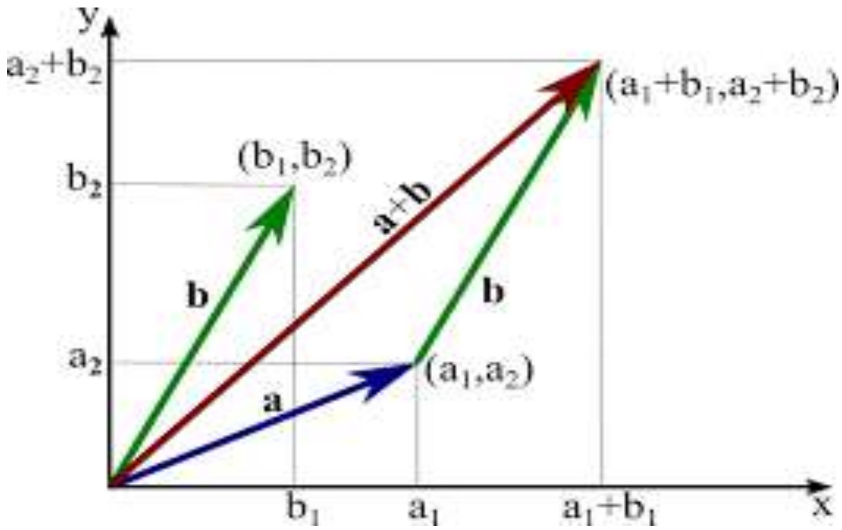
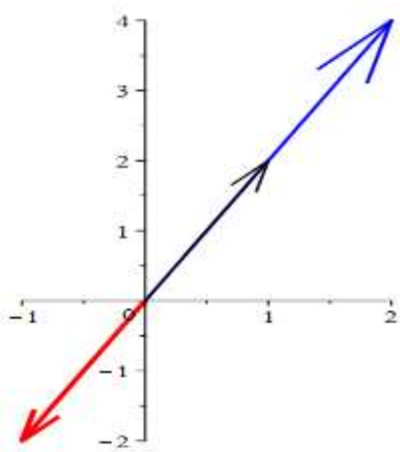
# Vector and Matrix Operations & Algebra

## Vector Addition




$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-2\\6\end{pmatrix},\quad \begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}-2\\4\\7\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}2\\-4\\-7\end{pmatrix}=\begin{pmatrix}3\\-2\\-4\end{pmatrix}$$

## Vector Multiplication by Constant

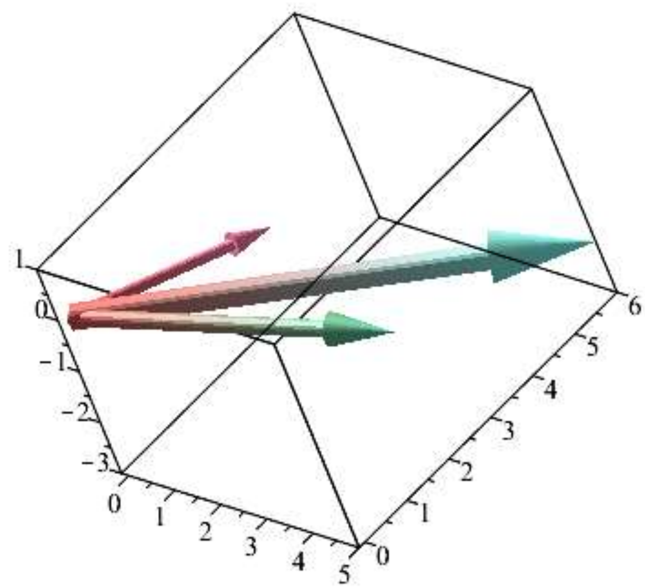


$$2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix},\quad -1\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\end{pmatrix}$$

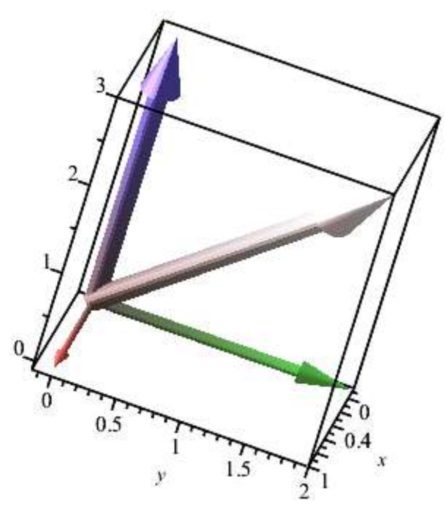
## Vectors in 3 Dimensions

## Vector Addition in 3 Dimensions

Unless the two vectors are on the same line, geometrically the sum of two vectors in $\mathbb{R}^3$ is the diagonal of a parallelogram on a plane in $\mathbb{R}^3$.




Unless the three vectors are on the same line, or on the same plane, geometrically the sum of three vectors in $\mathbb{R}^3$ is the diagonal of a parallelepiped in $\mathbb{R}^3$.



# Dot Product of Vectors

## Definition

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ be two vectors in $\mathbb{R}^n$ ($n-$dimensional vector space). Then

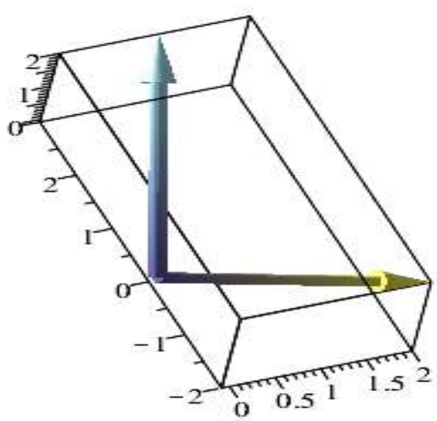
we define their **dot** product by

$$\vec{x}^{\,t}\,\vec{y} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n = \sum_{i=1}^{n} x_i \cdot y_i \, ,$$

where $\vec{x}^{\,t}$ is called the **transpose** of vector $\vec{x}$.

## Examples

**a.** $\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1(3) + 2(4) = 11$ **b.** $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2 + 4 - 6 = 0$ (see graph)



$$\textbf{c.}\quad \begin{pmatrix} 1 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 5 \\ 10 \end{pmatrix} = 1(1) + 0(4) + 1(5) - 1(10) = -4$$

# Norm of a Vector

## Definition

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a vector in $\mathbb{R}^n$. We define the Euclidean norm (length) of $\vec{x}$ as follows:

$$\| \vec{x} \|_2 = \sqrt{x_1^{\,2} + x_2^{\,2} + \cdots + x_n^{\,2}} = \sqrt{\sum_{i=1}^{n} x_i^{\,2}}$$

### Examples

Let $\vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 5 \\ 10 \\ -15 \end{pmatrix}$.

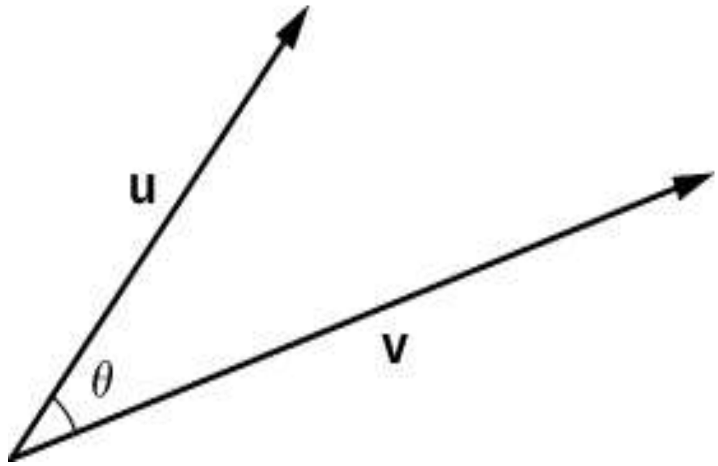
$$\| \vec{x} \|_2 = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\ ,$$

$$\| \vec{y} \|_2 = \sqrt{5^2 + 10^2 + (-15)^2} = \sqrt{25+100+225} = \sqrt{350} \simeq 18.7$$

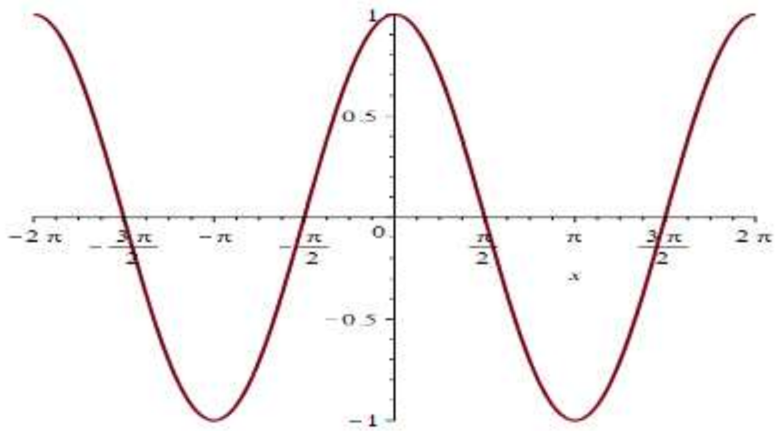
## Homework

Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -6 \\ -2 \\ 2 \end{pmatrix}$ are **orthogonal** (perpendicular). Two vectors $\vec{x}$ and $\vec{y}$ are perpendicular if $\vec{x}\,\vec{y} = 0$. What are their actual lengths?

## Geometry of a Dot Product




$$\vec{u}\;\vec{v} = \left\|\vec{u}\right\|_2 \cdot \left\|\vec{v}\right\|_2 \cdot \cos\theta$$




## Matrix and Vector Operations

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \pm \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix}, \qquad c \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -3 & 5 & -2 \end{bmatrix}_{3\times 3} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}_{3\times 1} = \begin{pmatrix} 1(3)+1(2)+1(0) \\ 1(3)+2(2)-1(0) \\ -3(3)+5(2)-2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ -1 & 6 \end{pmatrix}_{2\times 2} \begin{pmatrix} -2 & 5 \\ 1 & 10 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 3(-2)+4(1) & 3(5)+4(10) \\ -1(-2)+6(1) & -1(5)+6(10) \end{pmatrix} = \begin{pmatrix} -2 & 55 \\ 8 & 55 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ 0 & 1 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Remarks

a. Matrices in general do **not** commute. In other words, $AB \neq BA$.

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$, but $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$

b. $AB = 0$ does not imply that at least that one of the two matrices is the zero matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

c. $AB = AC$ does **not** necessarily imply that $B = C$.

d. Matrix division by a vector or another matrix is not defined.

e. $A(B + C) = AB + AC$.

f. $A(BC) = (AB)C$.

g. $k(AB) = (kA)B = A(kB)$ ( $k$: constant )

h. Two rectangular matrices $A_{m\times k}$ and $B_{n\times p}$ can be multiplied together as long as $k = n$.

## Homework

1. $$\begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & 4 & 4 & -1 \\ 5 & 3 & 6 & 10 \\ 0 & 2 & 0 & 2 \end{bmatrix}_{4\times 4} \begin{bmatrix} 1 & 1 \\ 5 & 4 \\ -2 & 0 \\ 0 & -3 \end{bmatrix}_{4\times 2} = \begin{bmatrix} -4 & -4 \\ 12 & 19 \\ 8 & -13 \\ 10 & 2 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} =$$

3. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & -3 \\ -1 & 1 & -2 \end{bmatrix} =$$

4. $$\begin{bmatrix} -4 & 5 \\ 2 & 10 \\ 6 & 20 \end{bmatrix}_{3x2} \begin{bmatrix} 0 & 8 \\ 11 & 13 \\ 1 & 0 \end{bmatrix}_{3x2} =$$

5. Find the angle between the vectors $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

6. Find the norm of the vectors $\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.